Non-Invertible Chiral Symmetry

Zekai Wang

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1 Introduction

We study the chiral symmetry in $3+1D$ massless QED, with the Lagrangian

$$\frac{1}{\sqrt{g}}\mathcal{L} = i\psi_L^\dagger\bar{\sigma}^\mu(\partial_\mu - iA_\mu)\psi_L + i\psi_R^\dagger\sigma^\mu(\partial_\mu + iA_\mu)\psi_R - \frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu} \tag{1}$$

Classically, this theory has a chiral symmetry, namely the Lagrangian is invariant under the left spinor rotation $\psi_L \to e^{i\alpha}\psi_L, \psi_R \to \psi_R$. The Noether current of associated is $j_L^\mu = \psi_L^\dagger\bar{\sigma}^\mu\psi_L$. In QFT, the chiral symmetry has ABJ anomaly and the current j_L^μ is not conserved. Instead we have

$$\mathrm{d}J_L = \frac{1}{8\pi^2}F\wedge F, \qquad J_L = *j_{L\mu}dx^\mu. \tag{2}$$

Equation.(2) can be derived elegantly by considering the transformation of path integral measure with gauge invariant regularization. The Jacobian of left spinor rotation gives the divergence of the current [1].

Despite the divergent current, we can define some charge as usual.

$$Q_L(t) \equiv \int \mathrm{d}^3x j_L^0(t,\vec{x}) \tag{3}$$

We can compute the net change of Q_L from past to future infinity, using eq.(2)

$$Q_L(+\infty) - Q_L(-\infty) = \int_{-\infty}^{+\infty} dt\frac{dQ_L}{dt} = \int_{\mathcal{M}} \frac{1}{8\pi^2}F\wedge F. \tag{4}$$

Here, $\left[\frac{F}{2\pi}\right] \in H^2(\mathcal{M},\mathbb{Z})$ where $\mathcal{M}$ is the spacetime manifold. The right-hand side is topological and takes an integer value on any four-dimensional spin manifold [2]. If $\mathcal{M}$ is topologically trivial, the right-hand side is evaluated to zero. The chiral-symmetry selection rule holds in such special cases for local operators and S-matrix elements.

However, above arguments can not be generalized to any spacetime at least by two considerations. First, due to the study of strong gravity systems, the modern perspective states that one should expect a well-defined QFT on any non-trivial spacetime topology. If the spacetime has non-trivial 2-cycles, e.g. $\mathcal{M} \cong S^2\times S^2$ or T^4, the field strength can have non trivial flux around such 2-cycles $\mathcal{C}^2$.

$$\frac{1}{2\pi}\int_{\mathcal{C}^2} F \in \mathbb{Z}, \tag{5}$$

The right-hand side of eq.(4) can take non zero values. Second, even in flat spacetime, the correlation function of 't Hooft lines can also be non zero since inserting a 't Hooft line effectively creates a 2-cycle in spacetime. Take another perspective, eq.(4) implies Q_L charge is violated by Abelian instanton density on $\mathcal{M}$ which is zero in $\mathbb{R}^4$ but not in general.

The violation of abelian chiral symmetry is relatively mild and one therefore is attempted to formally redefine the current as

$$\tilde{J} = J_L - \frac{1}{8\pi^2}A\wedge F \tag{6}$$

Such that $d*\tilde{J} = 0$, where A is the 1-form electric gauge field. This redefined current is not gauge invariant and $F = dA$ valid globally only when spacetime has no closed 2-cycles and no line operator inserted.

2 Construction of Chiral Defect Operator $D_{1/k}$ by TQFT

The symmetry action give rise to topological operators of different co-dimensions. If the underlining current is conserved, the operator is protected from local deformation of the manifold that supports the symmetry operator, provided that the deformation does not across any other charged operators under this symmetry. In such sense the symmetry operator is topological. Consider a co-dimension one symmetry operator with conserved $U(1)$ current J that is anomaly free:

$$\mathrm{U}(\Sigma^3, \alpha) \equiv \exp\left[i\alpha\left(\int_{\Sigma^3} J\right)\right], \tag{7}$$

where Σ^3 is some three dimensional manifold that supports U. This operator in 4D generates zero form ordinary symmetry. Charged operators are local operators supported at a single point. A local operator $\mathcal{O}_q(x)$ with charge q would transform by a phase $e^{iq\alpha}$ when U move across x, which can be interpreted as symmetry action.

For anomalous current satisfies eq.(2), we would formally construct a gauge dependent operator

$$\mathrm{U}'(\Sigma, \alpha) \equiv \exp\left[i\alpha\left(\int_{\Sigma} \tilde{J}\right)\right] = \exp\left[i\alpha\left(\int_{\Sigma^3} J_L - \frac{1}{8\pi^2}\int_{\Sigma^3} A\wedge F\right)\right]. \tag{8}$$

Here $-\frac{i\alpha}{8\pi^2}\int_{\Sigma^3} A\wedge F$ is a Chern-Simons term on Σ^3 with level "α". Which only make sense when

1. $\alpha \in 2\pi\mathbb{Z}$
2. $\alpha \in 2\pi\mathbb{Q}$

The first case is trivial since it corresponds to a 2π times integer rotation of $U(1)$ phase. The second case should be understood as a "response" to the path integral on Σ^3 and the corresponding operator generated are non-invertable in the sense the operator $U^\dagger U$ has zero eigen values.

Consider the $U(1)_k, k\in\mathbb{Z}$ Chern-Simons theory on Σ^3 coupled to background field F:

$$Z^k[F] = \int [\mathcal{D}c]\exp\left[\int_{\Sigma^3}\left(\frac{i}{2\pi}c\wedge F + \frac{ik}{4\pi}c\wedge dc\right)\right] \tag{9}$$

If $\Sigma^3 \cong S^3$, we can evaluate the partition function by Gaussian integration

$$Z^k[S^3] = L_1 \exp\left[-\frac{i}{4\pi k}\int_{S^3} A\wedge F\right] \tag{10}$$

With some prefactor L_1 depending on functional determinant $\det'(ik\partial_\mu/4\pi)$ that needs to be regularized. Importantly, we found $Z^k[F]$, on simple manifold S^3, gives us the response we wish for $\alpha = 2\pi/k$ in eq.(8). We further take the path integral as a part of the definition on more general manifold Σ^3 and define chiral defect operator

$$D_{1/k}(\Sigma^3) \equiv U(\Sigma^3, 2\pi/k)Z^k[F] = \exp\left[\frac{2\pi i}{k}\int_{\Sigma^3} J_L\right]\int[\mathcal{D}c]\exp\left[\int_{\Sigma^3}\left(\frac{i}{2\pi}c\wedge F + \frac{ik}{4\pi}c\wedge dc\right)\right] \tag{11}$$

To replace ill-defined operator $U'(\Sigma^3, 2\pi/k)$.

With the help of the minimum $\mathbb{Z}_k$ TQFT $\mathcal{A}^{k,p}$, we can construct $D_{p/k}$ for general rational number $\alpha = 2\pi p/k$ [3–5]. The construction of chiral defect operator $D_{1/k}$ presented in this section, by activating a TQFT on Σ^3, is analogous to the construction of fractional quantum hall states in condensed matter physics [6].

3 Construction of $D_{1/k}$ by Gauging Magnetic $\mathbb{Z}_k^{(1)}$ Symmetry

The theory eq.(1) has a $U(1)$ magnetic 1-form global symmetry, which we will denote by $U(1)^1_m$. The charged operators under $U(1)^1_m$ are one-dimensional 't Hooft lines $T_q(l)$ with charge q. We assume no dynamical magnetic monopoles in our theory and 't Hooft lines are unbreakable. The conserved current is 2-form magnetic flux

$$J_m = \frac{1}{2\pi}F, \qquad dJ_m = 0. \tag{12}$$

The symmetry operators are generated by J_m supported by codimension 2 closed surfaces Σ^2, associated with a $U(1)$ element $e^{i\alpha}$

$$U_m(\Sigma^2, e^{i\alpha}) = \exp\left[i\alpha\int_{\Sigma^2} J_m\right] = \exp\left[\frac{i\alpha}{2\pi}\int_{\Sigma^2} F\right] \tag{13}$$

The symmetry action on 't Hooft lines are given

$$U_m(\Sigma^2, e^{i\alpha})T_q(l) = \begin{cases} e^{iq\alpha}T_q(l) & l, \Sigma^2 \text{ linked,} \\ 0 & l, \Sigma^2 \text{ not linked} \end{cases} \tag{14}$$

In order to gauge $U(1)^1_m$, the first step is to couple J_m to some background gauge field B in Lagrangian with a term $B \wedge J_m$ just as we gauge electromagnetic $U(1)$ symmetry by adding a term $A_\mu j^\mu_{em}$ first. B is locally a 2-form. Gauging $U(1)^1_m$ sets equivalence class among different 't Hooft lines with different phases, namely $T_q(l) \sim e^{i\alpha}T_q(l)$. To compare two 't Hooft lines supported by different l_1, l_2, we parallel transform one another using connection B

$$T'_q(l_2, \mathcal{M}^{12}) = \exp\left[iq\int_{\mathcal{M}_{12}} B\right] T_q(l_1). \tag{15}$$

Here $\mathcal{M}_{12}$ is a surface with boundary l_1, l_2. The gauge equivalence indicates the gauge transformation of $B \mapsto B - d\beta$.

$$U_m(\Sigma^2_2, e^{i\alpha_2})T'_q(l_2, \mathcal{M}^{12}) = \exp\left[iq\int_{\mathcal{M}^{12}} B - d\beta\right] U_m(\Sigma^2_1, e^{i\alpha_2})T_q(l_1). \tag{16}$$

By Stokes Theorem, we have

$$\int_{l_2}\beta = \alpha_2 \qquad \int_{l_1}\beta = \alpha_1 \tag{17}$$

Since $dF = 0$, under $U(1)^1_m$ gauge transformation, the coupling $B \wedge F$ term vanishes after integration by parts. However, the equation of motion for A becomes

$$d * F = \frac{e^2}{\pi}dB. \tag{18}$$

The electric 1-form current would not conserve if $dB \neq 0$. In fact $U(1)^1_m$ and $U(1)^1_e$ have a mixed 't Hooft anomaly and cannot be gauged simultaneously [7]. Instead, we can gauge a $\mathbb{Z}_k$ subgroup of $U(1)^1_m$ by introducing the following terms.

$$S \supseteq \int \frac{i}{2\pi}b \wedge F + \frac{ik}{2\pi}b \wedge dc + \frac{ik}{4\pi}b \wedge b. \tag{19}$$

Here $k \in \mathbb{Z}$. We change $B \to b$ and use lower cases to label new dynamical fields, the path integral sum over all b, c gauge configurations. c is a $U(1)^0$ gauge field with quantized flux $\frac{1}{2\pi}\int_{\Sigma^2} dc \in \mathbb{Z}$. Summing over the local variation of c sets $db = 0$ locally by equation of motion, and summing over flux level of c sets holonomy of b to be $\mathbb{Z}_k$-valued. On simple manifold with trivial topology, b has no flux level and we can use equation of motion $b = -F/k$ to integrate out b, and get

$$S' = S - \frac{2\pi i}{k}\int \frac{1}{8\pi^2}F \wedge F. \tag{20}$$

It is shown that on general manifold, gauging $\mathbb{Z}^1_k \subseteq U(1)^1_m$ is equivalent to shift the QED Lagrangian by a θ-angle [3].

$$\theta \longrightarrow \theta - \frac{2\pi}{k} \tag{21}$$

In 3+1D massless QED, we can always do a fermion field redefinition to fully remove the θ-term thus gauging $\mathbb{Z}^1_k \subseteq U(1)^1_m$ leaves QED invariant, up to an isomorphism from chiral rotation. In fact, all theory with ABJ chiral anomaly is invariant under gauging $\mathbb{Z}^1_k \subseteq U(1)^1_m$, with appropriate torsion term corresponding to the last term in eq.(19).

Using the self-dual property of 3+1D QED, we can half gauge Z_k^1 over $x \geq 0$ to construct topological defect $D_{1/k}$ between the same(isomorphic) theory on surface $x = 0$. The explicit action is

$$S[b,c] = S_{\text{bulk}} + \int_{x\geq 0} \left(\frac{i}{2\pi} b \wedge F + \frac{ik}{2\pi} b \wedge dc + \frac{ik}{4\pi} b \wedge b \right) + \frac{2\pi i}{k} \int_{x=0} J_L + \frac{2\pi i}{k} \int_{x\geq 0} \frac{1}{8\pi^2} F \wedge F \quad (22)$$

The last two terms sum up to zero by anomaly equation. b, c are dynamical fields defined in $x \geq 0$. We choose Dirichlet boundary condition $b|_{x=0} = 0$. The equation of motion for b is $b + dc + F/k = 0$, integrating out b in the bulk and we get[1]

$$S[c] = S_{\text{bulk}} + \int_{x=0} \left(\frac{ik}{4\pi} c \wedge dc + \frac{i}{2\pi} c \wedge F + \frac{2\pi i}{k} J_L. \right) \quad (23)$$

Which is exactly an insertion of chiral defect operator $D_{1/k}$ at $x = 0$ surface.

4 Fusion rule of $D_{1/k} \times D_{1/k}^{\dagger}$

So far we have managed to construct some chiral defect operators supported on three-dimensional manifold Σ^3. These operators are topological and non-invertible in the sense that $DD^{\dagger} \neq 1$ in general. Applying eq.(11) we have

$$D_{1/k}(\Sigma^3) \times D_{1/k}^{\dagger}(\Sigma^3) = \int [DcD\bar{c}] \exp \left[\int_{\Sigma^3} \left(\frac{ik}{4\pi} (c \wedge dc - \bar{c} \wedge d\bar{c}) + \frac{i}{2\pi} (c - \bar{c}) \wedge F \right) \right] \equiv \mathcal{C}_k(\Sigma^3) \quad (24)$$

Where $\mathcal{C}_k(\Sigma^3)$ is the condensation defect operator from gauging $\mathbb{Z}_k^1 \subseteq U(1)_m^1$. Alternatively, the fusion rule is given by gauging $\mathbb{Z}_k^1 \subseteq U(1)_m^1$ over some finite region $\Sigma^3 \times I$, where two Dirichlet boundaries of I corresponds to $D_{1/k}$ and $D_{1/k}^{\dagger}$ respectively, which requires summing over all non-trivial b, c fields in eq.(19) but on $\Sigma^3 \times I$. Summing over c restricts the gauge field b in the second relative cohomology classes , namely

$$b \in H^2(\Sigma^3 \times I, \partial(\Sigma^3 \times I), \mathbb{Z}_k) \cong H_2(\Sigma^3 \times I, \mathbb{Z}_k) \cong H_2(\Sigma^3, \mathbb{Z}_k) \quad (25)$$

The first isomorphism comes from Lefschetz duality [3,8] and the second isomorphism is due to the trivial topology of I. Therefore, all the non-trivial gauge field configurations are characterized by surface $S \in H_2(\Sigma^3, \mathbb{Z}_k)$ with holonomy. Therefore, we have[2]

$$D_{1/k}(\Sigma^3) \times D_{1/k}^{\dagger}(\Sigma^3) = \frac{1}{k} \sum_{S \in H_2(\Sigma^3, \mathbb{Z}_k)} \exp \left(\frac{i}{k} \int_S F \right) \quad (26)$$

Where the constant in front is fix by standard normalization for $\mathbb{Z}_k^1$ gauge theory, which also related to L_1 in eq.(10).

The condensate $\mathcal{C}_k(\Sigma^3)$ lives on a three-dimensional manifold but is a sum over operators supported on two-dimensional surfaces S. The $U(1)_m^1$ generator $\frac{1}{2\pi} \int_S F$ acts trivially on local operators of the theory, therefore only $S = 0$ relevant and thw whole $\mathcal{C}_k$ acts trivially on local operators. Geometrically, $\mathcal{C}_k$ is a mesh on Σ^3 that local operators can freely move across.

With special topology, $\mathcal{C}_k$ can be zero. Suppose $\Sigma = S^2 \times S^1$ therefore only one 2-cycle on Σ and the flux is quantized. $\frac{1}{2\pi} \int_{S^2} F = m \in \mathbb{Z}$. The condensate operator can be decomposed into the direct sum of flux levels $\mathcal{C}_k = \oplus_m \mathcal{C}_k^m$ and

$$\mathcal{C}_k^m = \sum_{l=0}^{k-1} \exp \left(\frac{2\pi i l m}{k} \right) = \begin{cases} 1 & \text{if} \quad k|m \\ 0 & \text{else} \end{cases} \quad (27)$$

We found a large kernel for $D_{1/k} \times D_{1/k}^{\dagger}$, therefore reinstated the non-invertible nature of chiral defect operators.

[1] For general construction on any Σ^3 involves summing over flux levels of b, details presented in [3]

[2] Lefschetz duality will reduce the volume integral into a surface integral over S, with the price of dropping one b in the integrand, namely $\frac{i}{2\pi} \int_{\Sigma^3 \times I} b \wedge F = \frac{i}{k} \int_S F$.

5 Selection Rules

Symmetry of the theory often imposes selection rules on correlation functions and S-matrix elements. If the theory admits some ordinary global symmetry, the net charge of all charged operators appear in some non-trivial correlation function must be zero. For non-invertible symmetries, the selection rules have richer content due to the fact that symmetry defects can change the nature of operators.

We have shown in last section that the condensate operator $\mathcal{C}_k$ acts trivially on local operators. In fact the same holds for chiral defect operator $D_{1/k}$. The local operators do not interact with TQFT on Σ^3 thus only see $U(\Sigma^3, 2\pi/k)$ and equivalently perform a chiral rotation. With the language of half-gauging the $\mathbb{Z}_k^1$ magnetic 1-form symmetry, this is evident based on the self-dual property. However, the fermion mass operator would transforms under $D_{1/k}$

$$D_{1/k}(\Sigma^3)\left(m\bar{\Psi}\Psi(x)\right) = \bar{\Psi}\left(m\cos\frac{2\pi}{k} + im\gamma^5\sin\frac{2\pi}{k}\right)\Psi(x), \tag{28}$$

we can say electron is naturally massless in QED because of non-invertible chiral symmetry. Similar arguments can be applied to standard model and explain neutrino physics [9].

It is more interesting to see how chiral defect operator acts on 't Hooft lines. A 't Hooft line $T_q(l)$ can be viewed as the world line l of an infinite heavy magnetic monopole with magnetic charge q. When crossing $D_{1/k}$, the theory shifts its θ-angle by $2\pi/k$, thus implies the magnetic monopole would acquire an electric charge through Witten effect [10]. The equation of motion for electric current with the presence of θ term is $\frac{1}{e^2}d * F + \frac{\theta}{4\pi^2}F = dJ_{el}$. Therefore the electric charge quantization becomes

$$\int_\Sigma \left(\frac{1}{e^2} * F + \frac{\theta}{4\pi^2}F\right) \in \mathbb{Z} \tag{29}$$

For a 't Hooft line with no electric charge, Crossing $D_{1/k}$ obtains an electric charge $Q = q/k$. Therefore $D_{1/k}$ acts on $T_q(l)$ by

$$D_{1/k}(\Sigma^3)T_q(l) = T_q(l)\exp\left[i\int_{\Sigma'^3}\frac{F}{k}\right] \text{``}=\text{''} T_q(l)W_q^{1/k}(l, \Sigma'^3) \tag{30}$$

Here $\partial\Sigma'^3 = l$ the quotation mark means on the right hand side, the k-th root of Wilson line is inappropriately normalized thus not only depend on l but also on the open surface Σ'^3 bounded by l, the bulk deformation of the open surface is irrelevant.

The chiral defect operator $D_{1/k}$ turns a line operator into a surface operator, which is a general property of non-invertible symmetry operators. Therefore $D_{1/k}$ relates correlation function of different type of operators. For example, consider a correlator of local operator $\mathcal{O}_p$ and 't Hooft line T_q on S^4, which is a usual IR regulated $\mathbb{R}^4$, the non-invertible chiral symmetry imposes the following condition.

$$\langle\mathcal{O}_p(0)T_q(l)\rangle = e^{ip\alpha}\langle\mathcal{O}_pT_q(l)W_q^{1/k}(l, \Sigma'^3)\rangle. \tag{31}$$

6 Summary

In this paper, we analyzed chiral anomaly in 3+1D QED and constructed a non-invertible symmetry defect operator $D_{1/k}$ as an response of fractional quantum hall states. we further discussed half gauging $\mathbb{Z}_k$ subgroup of magnetic 1-form symmetry and give a equivalent construction of $D_{1/k}$. Using half-gauging, we discussed $D_{1/k} \times D^\dagger_{1/k}$ fusion rule and showed its non-invertible nature. Finally we discussed the selection rule of non-invertible chiral operator and its application to correlation function and explain the naturalness of electron mass.

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