Chern-Simons Theory and its relation to Jones polynomial and CFT

Zekai Wang

May 7, 2025

1 Chern-Simons Action in 2+1 Dimension

We want to formulate a generally covariant theory in which all observables will be topological invariants. Given a gauge group G, gauge 1-form connection A and an orientable 3-Manifold M, we can construct the following Chern-Simons action.

$$S_{\text{CS}} = \frac{k}{4\pi}\int_M \text{tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right). \tag{1}$$

The Chern-Simons action does not depend on any metric, which is different from Yang-Mills action that used to describe strong and weak dynamics in standard model of particle physics. We will consider the quantum field theory defined by non-abelian S_{CS} and discuss its relation to three dimensional geometry and two dimensional conformal field theory.

For any simple, compact Lie group G, the continuous map $M \to G$ is not connected especially $\pi_3(G) \simeq \mathbb{Z}$. For simplicity, the gauge group G is taken to be $SU(N)$ throughout this paper. Under large gauge transformations labeled by non-zero "winding number" m, the action is not invariant but has the following transformation law

$$S_{CS} \to S_{CS} + 2\pi km. \tag{2}$$

The gauge transformation should be the redundancy of our theory and do not affect all physics. Luckily, quantum field theory formulated by Feynman path integral only talks to the phase e^{iS}. which is invariant under large gauge transformation when

$$k \in \mathbb{Z}. \tag{3}$$

This gives a quantization condition on the parameter k, which is called the "level" of Chern-Simons theory.

2 WZW Current Algebra

The integer parameter k is closely related to the central charge of 2D WZW conformal field theory with current algebra [1, 2]

$$J^a(z)J^b(w) \sim \frac{k\delta^{ab}}{(z-w)^2} + \frac{if_c^{ab}J^c(w)}{z-w}. \tag{4}$$

Here $J^a(z)$ are $su(N)$ symmetry current, f_c^{ab} are Lie algebra structure constant and k is the "level" of affine Kac-Moody algebra $su(N)_k$. Wiggle denotes all singular part of JJ OPE. The stress energy tensor can be obtained via Sugawara construction:

$$T(z) = \frac{1}{2(k + C_2(G)/2)} : J^a J^a : (z), \tag{5}$$

where quadratic Casimir is normalized so that $C_2(SU(N)) = 2N$. Computing TT OPE with proper normal ordering that pick out only regular part, we have

$$T(z)T(0) \sim \frac{c}{2z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z}, \qquad c = \frac{k\dim(g)}{k + C_2(G)/2}, \tag{6}$$

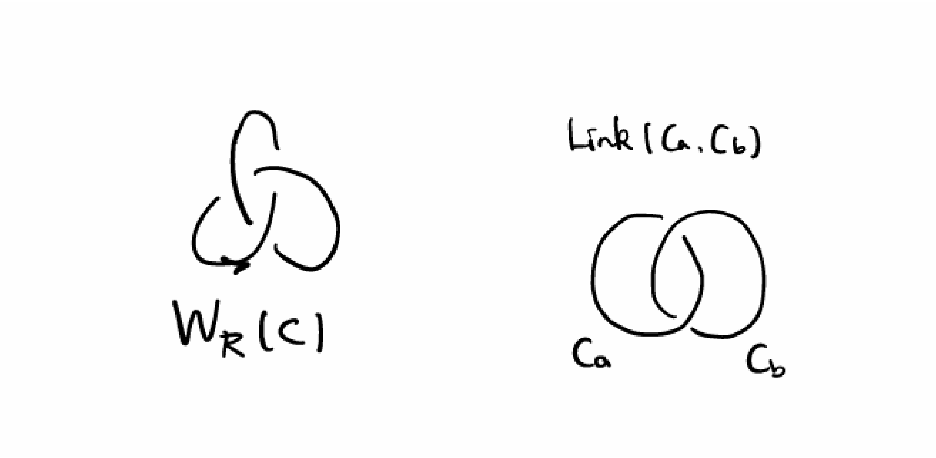

Figure 1: Wilson Loops and Links

which defines Virasoro Algebra with central charge c.

The shift from k to $k+C_2(G)/2$ in normalization constant of T is the effect of quantum correction. In large k limit of Chern-Simons theory, one can integrate the partition function $Z = \int \mathcal{D}A \exp(iS_{CS})$ with saddle point approximation: find the stationary points which are "flat connections $A^{(\alpha)}$" that satisfy $F_{ij}^{(\alpha)} = 0$, expand connection around these stationary points, $A_i = A_i^{(\alpha)} + B_i$, then integrate the fluctuation field B_i. Under gauge invariant regularization scheme one can deduce the perturbative partition function

$$Z = e^{i\pi\eta(0)/2} \cdot \sum_{\alpha} e^{i(k+C_2(G)/2)I(A(\alpha))} \cdot T_\alpha. \tag{7}$$

Here $I(A^{(\alpha)})$ is the Chern-Simons action evaluated at stationary points. And the front coefficients has been replaced by $k+C_2(G)/2$, which is in correspondence with WZW current algebra calculation, indicating the connection between two theories.

All the primary operators in WZW conformal field theory can be classified under irreducible representations of G. The conformal weight of such primary operators can be obtained by acting on the generator L_0 of Virasoro algebra constructed from eq.(5). The weight is given by

$$h_R = \frac{C_2(R)}{2(N+k)}. \tag{8}$$

3 Wilson Loop as Knots and Links

Chern-Simons theory is a pure gauge theory, there's no matter fields. Therefore, the physical observables should be Wilson lines, or more precisely, Wilson loops without end points. Given an oriented closed curve embed in 3D spacetime $S^1 \to M$. Wilson loops are characterized by irreducible representation R of the gauge group. We define

$$W_R(C) \equiv \mathrm{Tr}_R P \exp\left[i \int_C A\right]. \tag{9}$$

Here trace is taken in representation R. The exponential is defined by Taylor series and path-ordering P, taking account the non-commutativity of the Lie algebra. We notice that eq.(9) is independent of the metric, therefore Wilson loops are topological observables of Chern-Simons theory.

If we have multiple Wilson loop operators and we want to compute their expectation value, formally we can write out the Feynman path integral

$$\left\langle \prod_{i=1}^{r} W_{R_i}(C_i) \right\rangle \equiv \int \mathcal{D}A \exp\left[iS_{CS}\right] \prod_{i=1}^{r} W_{R_i}(C_i). \tag{10}$$

We want to show that when $M = S^3$ this expectation value gives knot invariants in 3D Euclidean space. For special case when gauge group is $SU(2)$, the corresponding invariants are Jones polynomial.

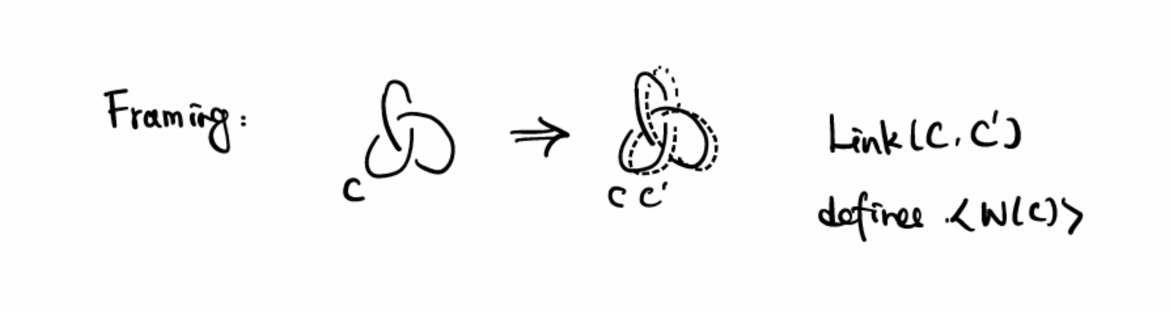

Figure 2: Framing regularization and self linking number.

Some clarification are needed when computing expectation value of multiple Wilson loops. For demonstration process we take the gauge group to be $U(1)$, technically this trivial case is out of our interest. When two loops linked together without crossing, one can explicitly evaluate the path integral in case of $r = 2$ [3]

$$\Big\langle W_a(C_a)W_b(C_b)\Big\rangle = \exp\left(\frac{i}{2k}n_a n_b \int_{C_a} dx^i \int_{C_b} dy^j \varepsilon_{ijk}\frac{(x-y)^k}{|x-y|^3}\right). \tag{11}$$

Here n_a, n_b are integers label representations of $U(1)$. x^i, y^j are coordinates on C_a, C_b respectively. When $x \neq y$, the double integral computes a well defined integer called Gauss linking number $\mathrm{Link}(C_a, C_b)$.

The right hand side of fig.(1) shows two loops with linking number 1, illustrating the fact that the Chern-Simons theory does lead to topological invariants as we hope. When $x = y$, the expression is ill-defined because the cubic term on the denominator diverges. This also happens when computing the expectation value of a single Wilson loop.

Such divergence is related to the fact that in 3D knot theory, there is no natural and topologically invariant way to regularize the self-linking number of a knot. Here we use "framing" to regularize the self-linking integral as shown in fig.(2). For each loop C we can assign a normal vector field $\mathcal{V}_C$ and deform the loop infinitesimally along $\mathcal{V}_C$ and get a new loop $C' : x^i \to x^i + \epsilon\mathcal{V}_C(x^i)$. The self-linking number is defined by the linking number between C and C'

$$\Big\langle W_R(C)\Big\rangle \equiv \mathrm{Link}(C, C'). \tag{12}$$

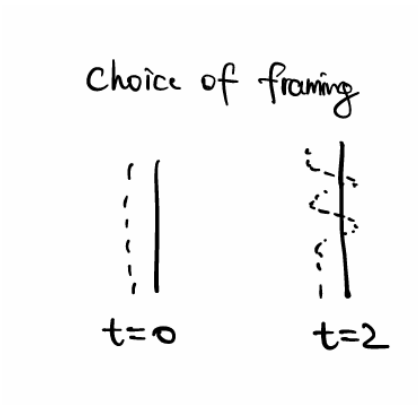

Figure 3

It is evident that the self-linking number defined this way depends not on the detail of vector field $\mathcal{V}_C$ but only on the topological class of this field. Physically the role of framing is similar to point-splitting regularization. However, on general 3-Manifold there is no canonical choice of framing. In particular we can change the linking number by twisting the frame, we state a general rule for how expectation values of Wilson loops change under a change of the framing: if we shift the framing of

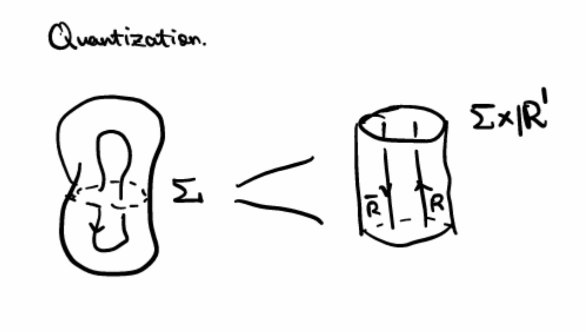

Figure 4: Quantization

the link C by t units, its self-linking number is increased by t, and the partition function is shifted by a phase

$$\left\langle W_R(C) \right\rangle_t = e^{2\pi i t h_R} \left\langle W_R(C) \right\rangle, \tag{13}$$

where h_R is the conformal weight of the primary field in R representation of WZW current algebra.

4 Quantization

So far we have discussed Chern-Simons theory in the language of Feynman path integral. In this section we will state a geometrical quantization procedure and show that the Hilbert space $\mathcal{H}$ is finite dimensional.

We cut our orientable 3 manifold M along a Riemann surface Σ. near the cut, M looks like $\Sigma \times R^1$. Canonical quantization on $\Sigma \times R^1$ will produce a Hilbert space $\mathcal{H}_\Sigma$ which is the physical Hilbert space of the Chern-Simons theory quantized on Σ. In particular Σ is oriented and if one reverses orientation of Σ, the Hilbert space must be replaced with its dual. More generally, we should consider the theory that has Wilson loops inserted, in which possible Wilson loops are "cut" by Σ. In this case Σ is presented with finitely many mark points $\{P_i\}$ with a G representation R_i assigned to each P_i. If one reverses the orientation of the Wilson line, the representation inserted should be replaced by its dual representation $\overline{R}$.

The canonical quantization formalism allows us to choose temporal gauge $A_0 = 0$ on Σ and the Chern-Simons action reduces to a first order derivative Lagrangian $\mathcal{L}_\Sigma$ integrated over "time" direction R^1 where

$$\mathcal{L}_\Sigma = \frac{k}{8\pi} \int_\Sigma \varepsilon^{ij} \mathrm{Tr} \left(A_i \frac{d}{dt} A_j \right), \tag{14}$$

together with a non-linear constraint equation

$$\varepsilon^{ij} F^a_{ij} = 0. \tag{15}$$

In perturbative quantum field theory one separates the canonical variables into "positions" $A_i(x)$ and "momenta" $\Pi_i(x)$. Then promote Poisson brackets into commutation relation of the corresponding quantum operators, which lead to the infinite-dimensional Hilbert space $\mathcal{H}_F$ of perturbative quanta. The constraints are imposed on $\mathcal{H}_F$. The situation here is different because the Lagrangian is already first-order in time derivative and there is no natural separation of A_i and Π_i. Instead, one first impose the constraint eq.(15) over "classical" canonical variables. The "classical" phase space obtained is the moduli space $\mathcal{M}$ of flat connections on Σ, modulo gauge transformations. Such moduli space are characterized by the holonomies around non-contractible loops. On a Riemann surface of genus $g > 1$ without any Wilson line insertion, $\mathcal{M}$ has dimension $(2g-2)d$ where d is the dimension of the gauge group. [4]

It can be shown ([5] [6]) that the dimension of the Hilbert space $\mathcal{H}_\Sigma$ is the same as the dimension of the space of the "conformal blocks" $\hat{\mathcal{H}}_\Sigma$. The association $\Sigma \to \hat{\mathcal{H}}_\Sigma$ is called a "modular functor". The space of conformal blocks in 1+1 D $su(N)_k$ WZW conformal field theory are the quantum Hilbert spaces obtained by quantizing a 2+1 D Chern-Simons theory. When M is Riemann sphere where genus is zero the Ward identities uniquely determine all the conformal blocks for descendants of the identity, therefore dimension is 1. On a complex Riemann surface of genus equal or greater than 1 $\hat{\mathcal{H}}_\Sigma$ is a finite dimensional vector space.

When Wilson lines are inserted and cut by Σ, the associated Hilbert space $\mathcal{H}_{\Sigma,R_i,P_i}$ are the space of conformal blocks with respect to the corresponding local operator insertion. Let Σ be an oriented surface of genus zero, with operators $\mathcal{O}_i$ insertion at P_i in R_i representation. The correlation function

$$\left\langle \prod_i \mathcal{O}_i(P_i, R_i) \right\rangle \tag{16}$$

can be decomposed as a linear combination of conformal blocks. Global G Ward identity requires the correlation function should be G invariant. Therefore, the number of conformal blocks are the number of trivial representations in the tensor representation. We have

$$\mathcal{H}_{\Sigma,R_i,P_i} = \mathrm{Inv}(\otimes_i R_i). \tag{17}$$

The formula is correct at sufficient large k, for small k there are finite correction and the actual Hilbert space is a subspace of eq.(17).

We consider a special interest case where G is $SU(N)$ group, Σ is Riemann sphere, four external insertions are in fundamental and anti-fundamental representations $(N, N, \overline{N}, \overline{N})$ respectively, $k \geq 2$. Since

$$N \otimes \overline{N} = 1 \oplus \mathrm{Adj}, \tag{18}$$

there are 2 trivial representations in tensor representation, dim $\mathcal{H} = 2$.

5 Knot Invariants

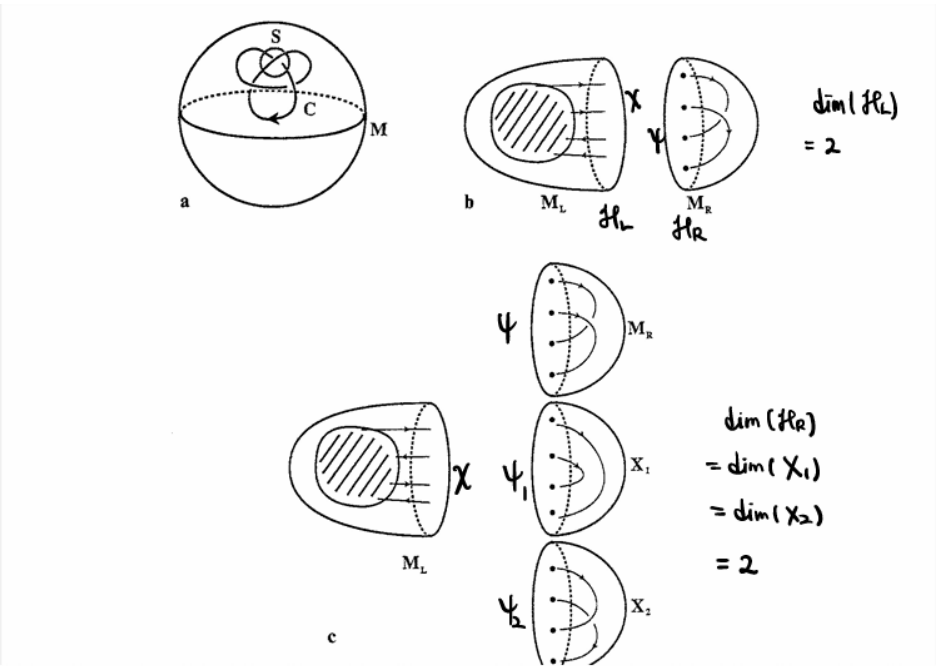

Figure 5: Expectation value as skein relation [7]

Consider a knot Wilson loop L in fundamental representation N on S^3, we want to compute $\langle W_N(L)\rangle$. Instead evaluating path integral eq.(10), we can draw a small sphere around the crossing as fig.(5), this cuts M into a complicated piece M_L with all other crossings that contain unknown information and a simple piece M_R, with four insertions on common boundary S^2. In M_R, these four insertions are connected by two lines. This is the special interest case we discussed in previous section where four insertions are in fundamental and anti-fundamental representation respectively. The Hilbert space $\mathcal{H}_L$ and $\mathcal{H}_R$ associated with the boundaries of M_L and M_R are two dimensional. We can construct two other Wilson loops L_1, L_2 that are identical to L in M_L region and different in M_R region by connecting four boundary insertion differently, as depicted. They correspond to three vectors $\{\psi, \psi_1, \psi_2\}$ in $\mathcal{H}_R$. Importantly, any three vectors in a two dimensional vector space must be linear dependent, thus we can always find three complex number α, β, γ such that

$$\alpha\psi + \beta\psi_1 + \gamma\psi_2 = 0. \tag{19}$$

Since L, L_1, L_2 are identical in M_L, they correspond to the same vector χ in $\mathcal{H}_L$. $\mathcal{H}_L$ and $\mathcal{H}_R$ are canonically dual and the partition function is equal to the natural inner product

$$\langle W_N(L)\rangle = (\chi, \psi). \tag{20}$$

Taking inner product with χ, eq.(19) becomes

$$\alpha\langle L\rangle + \beta\langle L_1\rangle + \gamma\langle L_2\rangle = 0. \tag{21}$$

From now on we use $\langle L\rangle$ as shorthand for $\langle W_N(L)\rangle$. eq.(21) is called skein relation in knot theory. Skein relation is a linear relation of knot invariants that can recursively determine those knot invariants of all knots. Jones polynomial can be defined via skein relation as follows. Assign a Laurent polynomial $V_L(t)$ to each knot (or link) with the following properties:

- Two equal knots (links) have the same polynomial.
- The polynomial of unknot V_O is 1.
- (Skein relation) If three knots L_+, L_0, L_- have pictures which are identical apart from within a region, then
$$\frac{1}{t}V_{L+} - tV_{L-} = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{L0}. \tag{22}$$

Figure 6: Skein relation

Suppose we want to compute a knot with p crossings. Locally if we change one over (under) crossing $L_+(L_-)$ to under (over) crossing $L_-(L_+)$, the resulting new knot should be no more complicated than the original one. On the other hand, L_0 is always way simpler and often transforms knots into links. Therefore, Jones polynomials should also assign to multiple links and the word "knots" also refers to links in this context. Inductively, suppose all knot invariants with at most $p-1$ crossings have already computed and the number of knots with p crossings are finite, we can use skein relation recursively and compute all knot invariants with p crossings.

Comparing eq.(21) and eq.(22), there are two caveats need attention:

- What's the relation between complex number α, β, γ and Jones polynomial argument t?
- Whether the expectation value of unknot $\langle O\rangle$ is 1?

We will address these two questions in next section.

6 Jones polynomial as expectation value

In order to determine parameters α, β, γ, we need to determine the explicit relation among three vectors ψ, ψ_1, ψ_2. The three configurations in M_R can transform into each other via "half-monodromy" diffeomorphism on S^2. We call this half-monodromy operation B. B exchanging two equivalent points on S^2 is sketched in part ***a*** of fig.(7). Two insertions move along the dashed line and changes projective

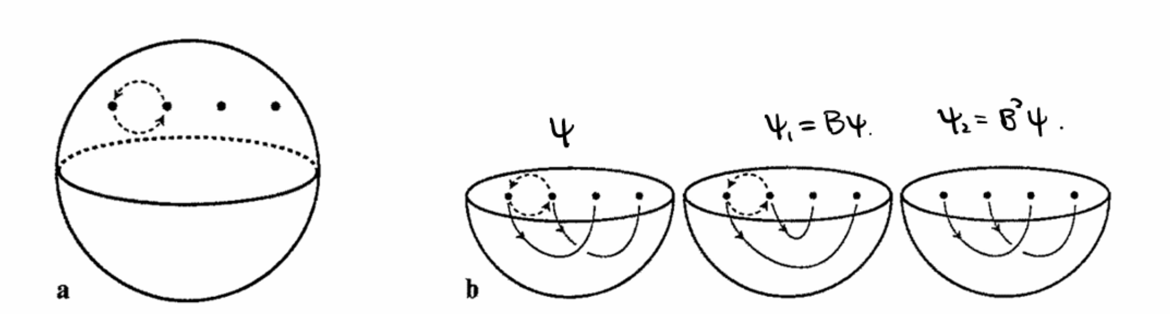

Figure 7: Half-monodromy operation on S^2 [7]

picture of the knot. Part b corresponds to three different breading pattern in M_R and they differ by a succession of B actions. The states ψ_1 and ψ_2 can be expressed as

$$\psi_1 = B\psi, \qquad \psi_2 = B^2\psi. \tag{23}$$

The operator B acting on a two-dimensional vector space is a two by two matrix, thus obeys a characteristic equation

$$B^2 - yB + z = 0, \tag{24}$$

where

$$y = \mathrm{Tr}(B), \qquad z = \det(B). \tag{25}$$

Acting on vector ψ, the characteristic equation becomes

$$z\psi - y\psi_1 + \psi_2 = 0. \tag{26}$$

Let h_R be the conformal weight of a primary conformal field transforming as R, let E_i be the irreducible representations of $SU(N)$ appearing in the decomposition of $R \otimes R$, and let h_{E_i} be the weights of the corresponding primary fields. Then the eigenvalues of B are [6]

$$\lambda_i = \pm \exp\left(i\pi(2h_R - h_{E_i})\right), \tag{27}$$

where plus or minus sign corresponds to whether E_i appears symmetrically or anti-symmetrically in tensor representation. In our case, R is the fundamental representation N of $SU(N)$, and

$$N \otimes N = \mathrm{Sym}^2(N) \oplus \mathrm{Alt}^2(N). \tag{28}$$

The tensor product of two fundamental representation can be decomposed as the sum of symmetric and antisymmetric representations. The conformal weight of corresponding primary fields are

$$h_N = \frac{N^2 - 1}{2N(N+k)}, \quad h_{\mathrm{Sym}} = \frac{N^2 + N - 2}{N(N+k)}, \quad h_{\mathrm{Asym}} = \frac{N^2 - N - 2}{N(N+k)}. \tag{29}$$

Therefore, the eigenvalues of B are

$$\lambda_1 = \exp\left(\frac{i\pi(-N+1)}{N(N+k)}\right), \qquad \lambda_2 = -\exp\left(\frac{i\pi(N+1)}{N(N+k)}\right). \tag{30}$$

In section 3, we have discussed the regularization scheme in defining the expectation value of a single knot. Under the half-monodromy action on M_R, the framing of the knot is twisted by 1 unit. We must incorporate this twist correction in calculation of α, β, γ by doing the following substitution,

$$\alpha \to \alpha, \quad \beta \to \beta e^{-2\pi i h_R}, \quad \gamma \to \gamma e^{-4\pi i h_R}. \tag{31}$$

One can furthermore introduce the variable

$$t = \exp\left(\frac{2\pi i}{N+k}\right), \tag{32}$$

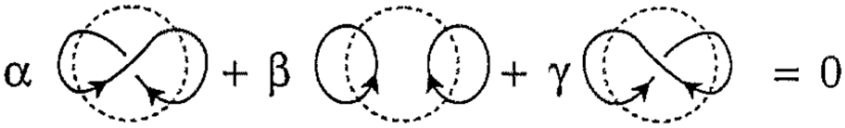

Figure 8: Skein relation for unknot [7]

Up to an common factor, the skein relation eq.(21) can be written as

$$-t^{N/2}\langle L_+\rangle + (t^{1/2} - t^{-1/2})\langle L_0\rangle + t^{-N/2}\langle L_-\rangle = 0. \tag{33}$$

The knot invariant defined by skein relation eq.(33) is called HOMEFLY invariant. Compared with eq.(22), When $N = 2$ we have:

> The expectation value of Wilson loops in level k, $SU(2)$ Chern-Simons theory on S^3 with standard framing regularization gives Jones polynomial of argument $t = \exp(2\pi i/(2 + k))$. [7]

Finally we could answer the question at the end of section 5. Consider the skein relation in fig.(8), L_+, L_- are both isotopic to a single unknot and L_0 is isotopic to two unknots without linking. Therefore

$$(\alpha + \gamma)\langle O\rangle + \beta\langle O^2\rangle = 0. \tag{34}$$

Using the same analysis as in section 5, we can chop $\langle O^2\rangle$ into two pieces with one unknot on each side. The boundary has no Wilson lines cross and the Hilbert space is one-dimensional. We can deduce that

$$\langle O^2\rangle = \langle O\rangle^2. \tag{35}$$

Therefore the expectation value of an unknotted Wilson loop is given by

$$\langle O\rangle = -\frac{\alpha + \gamma}{\beta} = \frac{t^{N/2} - t^{-N/2}}{t^{1/2} - t^{-1/2}} \tag{36}$$

We have substituted the value of α, β, γ in the second equation. When $N = 2$, the expectation value of unknot is not equal to 1, instead

$$\langle O\rangle = t + t^{-1}. \tag{37}$$

7 Final remarks

In the past few decades, the interplay between Chern-Simons theory, knot theory, and conformal field theory has profoundly shaped both mathematics and theoretical physics. [8,9] The Reshetikhin-Turaev construction is the FQFT construction of a 3d TQFT from the data of a modular tensor category. The RT-construction for group G is expected to be the FQFT of G-Chern-Simons theory, the Fuchs-Runkel-Schweigert-construction builds from the RT-construction explicitly the rational 2-dimensional 2d CFT boundary theory. Together, these fields continue to illuminate the rich interface between topology, geometry, and quantum physics, fostering ongoing advances in both mathematical understanding and physical applications.

References

[1] V.G. Knizhnik and A.B. Zamolodchikov. Current algebra and Wess-Zumino model in two dimensions. *Nuclear Physics B*, 247(1):83–103, December 1984.

[2] Philippe Di Francesco, Pierre Mathieu, and David Sénéchal. *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer New York, New York, NY, 1997.

[3] Alexander M. Polyakov. Fermi-Bose Transmutations Induced by Gauge Fields. *Mod. Phys. Lett. A*, 3:325, 1988.

[4] M. F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. *Phil. Trans. Roy. Soc. Lond. A*, 308:523–615, 1982.

[5] Erik P. Verlinde. Fusion Rules and Modular Transformations in 2D Conformal Field Theory. *Nucl. Phys. B*, 300:360–376, 1988.

[6] Gregory W. Moore and Nathan Seiberg. Polynomial Equations for Rational Conformal Field Theories. *Phys. Lett. B*, 212:451–460, 1988.

[7] Edward Witten. Quantum Field Theory and the Jones Polynomial. *Commun. Math. Phys.*, 121:351–399, 1989.

[8] N. Reshetikhin and V. G. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Inventiones mathematicae*, 103(1):547–597, December 1991.

[9] VLADIMIR G. TURAEV. Modular categories and 3-manifold invariants. *International Journal of Modern Physics B*, 06(11n12):1807–1824, 1992.

[10] J. Frohlich and C. King. The Chern-Simons Theory and Knot Polynomials. *Commun. Math. Phys.*, 126:167, 1989.