Higher Form Symmetry for Baby

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1 Introduction

Recent years have witnessed theoretical physicists' discovery of new forms of symmetries, usually referred to as *generalized symmetry.* With the foundational paper by Gaiotto, Kapustin, Seiberg & Willett in 2014. Some examples of generalized symmetry are *subsystem symmetries, higher-form symmetries and non-intvertable symmetries.*

Higher-Form symmetries are present in many ordinary theories and focus on the study of the completely anti-symmetric conserved currents $J^{\mu\nu\cdots}$ which can be interpreted as the components of some differential form. Thus we can integrate over certain sub-dimensional manifolds, rather than in the whole space to construct conserved charges. The charged objects are no longer local operators, but are extended objects (lines, surfaces, and etc). Moreover, the mathematical language of differential forms reveal its topological nature.

2 Review of Ordinary Symmetry

2.1 Noether's Theorm

The Story of Symmetry began with the inspirational observation by Emmi Noether's Theorm of continuous global symmetry and conserved currents. Suppose the physical system is described by least action principle and the action has the integration form of $S = \int \mathrm{d}^{\mathrm{D}}x\mathcal{L}[\phi]$ where ϕ is some general field content. The infinitesimal symmetry acting on the field is

$$\phi \longrightarrow \phi + \epsilon_a(x)\delta\phi_a. \tag{1}$$

Here ϵ_a are some infinitesimal parameters and we also promote to local symmetries (with coordinate dependence). Since the global subdivision of the symmetry keeps action invariant, one can assure the derivation of the action takes the form of

$$\delta S = \int \mathrm{d}^{\mathrm{D}}x J^{\mu}_a \partial_\mu \epsilon_a(x). \tag{2}$$

When $\epsilon_a(x) = \epsilon_a$ eq.(2) vanishes, indicating the global symmetry aspect. On the other hand any variation of action should be zero when field content satisfy the equation of motion. Therefore eq.(2) is identically zero with on-shell condition. By performing integration by parts and using the arbitrariness of $\epsilon_a(x)$, one can deduce the conservation law of currents

$$\delta S = \int \mathrm{d}^{\mathrm{D}}x \partial_\mu J^{\mu}_a \epsilon_a(x) = 0 \quad \Rightarrow \quad \partial_\mu J^{\mu}_a = 0. \qquad \text{(On-Shell)} \tag{3}$$

The charge Q_a is defined by integrating over all space slices in Lorentzian signature field theory.

$$Q_a \equiv \int \mathrm{d}^{\mathrm{D}-1}x J^{0}_a \tag{4}$$

Note that the current J^{μ}_a is not uniquely defined. We can always introduce an arbitrary anti-symmetric current rank-2 tensor $\Omega^{\mu\nu}_a$ and define

$$\tilde{J}^{\mu}_a \equiv J^{\mu}_a + \partial_\nu \Omega^{\mu\nu}_a, \tag{5}$$

Which is also conserved and leads to the same charge Q_a.

2.2 Ward Identity

In path integral formalism of quantum theory, the central quantity of interest is the correlation function of some operator

$$\langle X\rangle \equiv \int \mathcal{D}\phi X e^{iS(\phi)}. \tag{6}$$

Here $X \equiv \prod_j \phi(x_j)$ is some generic products of field operators. Suppose we do a symmetry transformation $\phi \to \phi'$ as eq.(1). Since the field operators are dummy indices of functional integral, we have

$$\begin{aligned}
\langle X\rangle &= \int \mathcal{D}\phi' X' e^{iS'(\phi)} &(7)\\
&= \int \mathcal{D}\phi \left(\mathrm{Det}\frac{\partial\phi'}{\partial\phi}\right)\left(X + \sum_j \phi(x_1)\dots\delta\phi_j(x_j)\dots\phi(x_N)\right) e^{i(S+\delta S)} &(8)\\
&= \int \mathcal{D}\phi\,(1+i\delta A)\left(X + \sum_j \phi(x_1)\dots\delta\phi_j(x_j)\dots\phi(x_N)\right)(1+i\delta S))\,e^{iS} &(9)\\
&= \langle X\rangle + i\langle \delta A X\rangle + i\langle \delta S X\rangle + \sum_j \langle \phi(x_1)\dots\delta\phi_j(x_j)\dots\phi(x_N)\rangle. &(10)
\end{aligned}$$

If we parameterize the possible anomaly operator as $\delta A = \int d^D x \mathcal{O}_a \epsilon_a(x)$ and we already have $\delta S = \int \mathrm{d}^{\mathrm{D}} x J^\mu_a \partial_\mu \epsilon_a(x)$. Then (In the absense of anomalies) the above relation leads to the *Ward identities*

$$\partial_\mu \langle J^\mu_a X\rangle = \langle \mathcal{O}_a(x) X\rangle - i\sum_j \delta(x-x_j)\langle \phi(x_1)\dots\delta_a\phi_j(x_j)\dots\phi(x_N)\rangle \tag{11}$$

Assuming X is a single field $\phi(y)$ and no anomalies, we have

$$\partial_\mu \langle J^\mu_a(x)\phi(y)\rangle = -i\delta(x-y)\langle \delta_a\phi(y)\rangle \tag{12}$$

By integrating x over $\mathcal{V} \equiv [y^0-\epsilon, y^0+\epsilon] \times \boldsymbol{R}^{n-1}$, we get

$$\langle Q(y^0+\epsilon)\phi(y)\rangle - \langle \phi(y) Q(y^0-\epsilon)\rangle = -i\langle \delta_a\phi(y)\rangle, \tag{13}$$

because of the time-ordering inherent to the path integral. In the limit of $\epsilon \to 0$ the left hand side is identified with the equal time commutator

$$\langle [Q_a, \phi(y)]\rangle = -i\langle \delta_a \phi(y)\rangle \tag{14}$$

Which corresponds to the infinitesimal transformation of operator under symmetry action. We say the field $\phi(x)$ is charged under the symmetry with charge Q_a and the finite transformation law is generated by *Lie algebra* with generators Q_a. For a group element g and the representation R under which ϕ furnish symmetry,

$$\phi \to \phi' = U(g)\phi U(g)^\dagger = R(g)\phi, \tag{15}$$

and

$$U = e^{i\theta_a Q_a} \qquad R(g) = e^{\theta_a T_a}. \tag{16}$$

For infinitesimal parameter θ_a this recovers eq.(14) with the identification $\delta_a\phi \equiv T_a\phi$

3 Rephrasing Symmetry in terms of Topology

3.1 Notation for Differential Form

In studying the theory of higher form symmetry, it is useful to use the notation of *differential forms*. Essentially, a differential p-form is an anti-symmetric rank p tensor and can be integrated over a p-dimensional manifold. This notation can make it easier to think about whether terms in an action are topological, meaning that they do not depend on the metric tensor.

Given a d-dimensional manifold with coordinates x, a 0-form is just a function $f(x)$. A 1-form is a vector contracted with a differential,

$$A \equiv A_\mu(x)dx^\mu. \tag{17}$$

This quantity can be integrated on a 1-dim sub-manifold (curve) γ. We denote as

$$\int_\gamma A = \int_\gamma A_\mu dx^\mu = \int_0^1 d\sigma \left(A_\mu \frac{dx^\mu}{d\sigma} \right). \tag{18}$$

Here σ is a parameterize of the curve γ. We find that this integral is independent of the way we parameterize it. It is a metric independent quantity.

Similarity, a 2-form should be a quantity such that it can be integrated on a 2-dimensional surface Σ. We define

$$F \equiv \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu, \quad dx^\mu \wedge dx^\nu \equiv dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu. \tag{19}$$

Here wedge $\wedge$ is defined to be an anti-symmetric tensor product of forms. Thus we can integrate a 2-form as

$$\int_\Sigma F = \int_\Sigma \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu. \tag{20}$$

In $d = 3$ case, this is the ordinary flux integral. To clarify this, we contract by Levi Ci-vita symbol in $\mathbb{R}^3$

$$\int_\Sigma \frac{1}{2}F_{ij}dx^i \wedge dx^j = \int_\Sigma B_k\epsilon^{ijk}dx^i \wedge dx^j = \int_\Sigma \vec{B} \cdot d\vec{S} \tag{21}$$

Following the path exploited above, the p-form is defined as

$$\omega_p \equiv \frac{1}{p!}\omega_{i_1\ldots i_p}dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \tag{22}$$

A p-form can wedge to a q-form and become a $(p + q)$-form. The anti-symmetric property requires

$$\omega_p \wedge \eta_q = (-1)^{pq}\eta_q \wedge \omega_p \tag{23}$$

The exterior derivative maps a p-form to a $(p + 1)$-form by

$$d\omega \equiv \frac{\partial \omega_{i_1\ldots i_p}}{\partial x^\mu}dx^\mu \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \tag{24}$$

For example $dA = F$ with components $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The anti-symmetry property implies that $d^2 = 0$, i.e.,

$$d(d\omega) = 0. \tag{25}$$

We call a form ω *closed* if $d\omega = 0$, and exact if there exist another form ζ such that $\omega = d\zeta$. In EM, field strength and potential have relations $F = dA$, which means F is an exact form. The homogeneous Maxwell equation $dF = 0$ further indicates F is a closed form. So exact forms are always closed but not in reverse. The space of closed forms modulo exact forms are *de Rham cohomology.*

The Hodge Star operation $\star$ is defined as

$$\star(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}) \equiv \frac{1}{(D-p)!}\epsilon^{\mu_1\cdots\mu_p}{}_{\mu_{p+1}\cdots\mu_D}dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_D} \tag{26}$$

Intuitively the Hodge star replace basis 1-forms with their complement 1-forms. For example in 3-d the Hodge star converts a 2-form into a 1-form $\star(dy \wedge dz) = dx$. Two Hodge star approximately returns to the original forms, up to a sign.

One of the triumph achievements of the differential form and manifold is *the Stokes's theorem.* Given a orientable p-dimensional manifold M with boundary ∂M and a $(p - 1)$-form ω_{p-1}, we have

$$\int_M d\omega_{p-1} = \int_{\partial M} \omega_{p-1}. \tag{27}$$

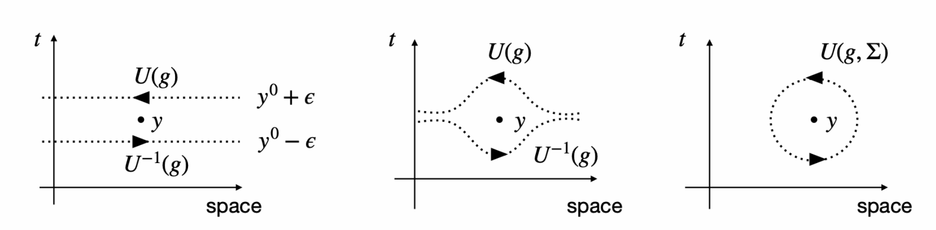

Figure 1: Deformation of the integral surface

3.2 Topological feature of charges

The Noether current can be thought as the components of a 1-form

$$J_a = J_a^\mu dx_\mu. \tag{28}$$

The conservation law can be written as

$$d \star J_a = 0 \tag{29}$$

In order to focus on the topological feature of the symmetry, we consider the Euclidean field theory, where time and space are treated on equal foot. Therefore, instead of integrating over the entire space of a time slice, we can integrate eq.(12) over a D-dimensional region Ω with boundary $\partial\Omega = \Sigma$ surrounding the point y.

$$\int_\Omega \langle d \star J_a \phi(y)\rangle = \int_\Sigma \langle \star J_a \phi(y)\rangle = -i \int_\Omega d^D x \delta(x-y)\langle \delta_a \phi\rangle. \tag{30}$$

We have used Stokes theorem in the first equality. The Noether charges are thus defined as an integral over a (D-1)-dimensional manifold Σ.

$$Q_a(\Sigma) \equiv \int_\Sigma \star J. \tag{31}$$

We also define *the link number* of the (D-1)-dimensional manifold Σ and the 0-dimensional manifold (point) y

$$\text{Link}(\Sigma, y) \equiv \int_\Omega d^D x \delta(x-y) \tag{32}$$

Here $Link(\Sigma, y) = 0$ or 1 depending on whether y is in Ω or not. With these settings we can write eq.(30) as

$$\langle Q_a(\Sigma)\phi(y)\rangle = -i\text{Link}(\Sigma, y)\langle \delta_a \phi(y)\rangle \tag{33}$$

The link number defined is clearly topological since it is unaffected by deformations of the surface Σ as long as do not cross the point y. Therefore the charge operators $Q_a(\Sigma)$ are also a topological invariant.

We can recover the canonical transformation of charged operator eq.(14) by choosing Σ as two different time slices of the entire space that connected at infinity distance. The space-time region contained can be continuously deformed into a closed sphere around point y. Therefore the operator $Q_a(\Sigma)$ defined in eq.(31) relates to the canonical charge operator Q_a by

$$\langle Q_a(\Sigma)\phi(y)\rangle = \langle [Q_a, \phi(y)]\rangle \tag{34}$$

We can also present the finite version of transformation under symmetry. Denote $U(g, \Sigma)$ as the unitary operator associated with symmetry group element g and surface Σ, therefore eq.(15) is rephrased as

$$\langle U(g, \Sigma)\phi(y)\rangle = R(g, \text{Link})\langle \phi(y)\rangle. \tag{35}$$

Here

$$U(g, \Sigma) = e^{i\theta_a Q_a(\Sigma)} \quad \text{and} \quad R(g, \text{Link}) = \begin{cases} R(g) & \text{Link} = 1 \\ 0 & \text{Link} = 0 \end{cases} \tag{36}$$

We refer to this as a 0-form symmetry, in the sense that the charged objects are local operators $\phi(y)$ supported in a 0-dimensional manifold (point). Moreover, the parameter of transformation is a closed 0-form, which is simply a constant θ_a.

The construction above gives us an illuminating perspective on symmetries in field theory, namely,

$$\text{Symmetry Generator} \Longleftrightarrow \text{Topological Operator.} \tag{37}$$

The charged operators corresponding to the objects with nontrivial link with the topological operator. This provides a way to generalize the notion of symmetry to the case of higher-forms. The essence of this generalization lies in the nontrivial link between higher dimensional objects.

4 Higher-Form symmetries

We generalize the discussion in the last part to 1-form symmetry. We expect the charged operator should be a 1-dimensional manifold (line) $W[\mathcal{C}]$ and the parameter is a closed 1-form

$$\xi = \xi_\mu dx^\mu. \tag{38}$$

From now on we omit group index a. We can define a 2-form current J to realize the 1-form transformation parameter,

$$J = \frac{1}{2} J_{\mu\nu} dx^\mu \wedge dx^\nu. \tag{39}$$

To clarify this, it is useful to reconsider the ordinary symmetry transformation eq.(2) in differential forms,

$$\delta S = \int_{\mathcal{M}^D} \star J \wedge d\epsilon. \tag{40}$$

Here ϵ is promoted to a local parameter, thus not closed. Therefore, if we want to generalize ϵ to a 1-form ξ, $\star J$ should be a $(D-2)$ form. equivalently, the conserved current J should be a 2-form. Now the conservation law $d \star J = 0$ reads

$$\partial_\mu J^{\mu\nu} = 0, \quad \text{with} \quad J^{\mu\nu} = -J^{\nu\mu}. \tag{41}$$

A note on the 1-form parameter ξ. The concept global transformation has been generalized to closed in our terminology. For a 1-form, it is translated into *flatness* condition on the parameter, namely,

$$d\xi = \partial_\nu \xi_\mu dx^\mu \wedge dx^\nu = 0 \quad \Leftrightarrow \quad \partial_\mu \xi_\nu - \partial_\nu \xi_\mu = 0. \tag{42}$$

The charge should be defined on a (D-2)-dimensional surface $\Sigma_{(D-2)}$ as

$$Q(\Sigma_{(D-2)}) \equiv \int_{\Sigma_{(D-2)}} \star J \tag{43}$$

For clarity, we call $Q(\Sigma)$ *symmetry operator* in order to distinguish form charged operator $W[\mathcal{C}]$.

4.1 Connecting Forms with Manifolds – Poincaré Duality

Rigorously, we need to prove that for a 1-form symmetry transformation θ, the charged objects are truly a 1-dimensional manifold $W[\mathcal{C}]$. The mystery lies in the mathematical theorem of *Poincaré Duality.* In a d-dimensional spatial slice, Poincaré duality provides a way to associate a (d-p)-form with a p-dimensional manifold.

$$\xi_{i_{p+1}\ldots i_d}(x) \equiv \int_{\Sigma_p} \epsilon_{i_1\ldots i_p i_{p+1}\ldots i_d} \delta^{(d)}(x-y) dy^{i_1} \wedge \cdots \wedge dy^{i_p}. \tag{44}$$

The parameter of the symmetry is then identified as the (d-p)-form ξ_{d-p} constructed from the submanifold Σ_p of dimension p. Poincaré duality automatically ensures the ξ_{d-p} constructed is closed, as long as the manifold Σ_p is closed, which is the condition for generalized version of "global" symmetry.

For ordinary symmetry, d=D-1 is the full space, and $p = d$ since the charge is the full spatial integration. Thus

$$\xi(x) = \frac{1}{d!} \int_{\Sigma_d} \epsilon_{i_1\ldots i_d} \delta^{(d)}(x-y) dy^{i_1} \wedge \cdots \wedge dy^{i_d} = 1. \tag{45}$$

This means that the parameters of ordinary symmetry are closed 0-forms. For 1-form symmetry $p = d - 1$, the Poincaré dual is a 1-form

$$\xi_{i_d}(x) = \frac{1}{(d-1)!}\int_{\Sigma_{d-1}} \epsilon_{i_1 \ldots i_{d-1} i_d} \delta^{(d)}(x-y) dy^{i_1} \wedge \cdots \wedge dy^{i_d - 1}. \tag{46}$$

The parameter of the transformation is

$$\int_{\mathcal{C}} \xi_1(\Sigma_{d-1}) = \int_{\mathcal{C}} \xi_i dx^i. \tag{47}$$

The infinitesimal transformation of a line operator reads

$$W[\mathcal{C}] \to W[\mathcal{C}]' = W[\mathcal{C}] + \int_{\mathcal{C}} \xi_1(\Sigma_{d-1}) \delta W[\mathcal{C}]. \tag{48}$$

4.2 Ward identities for 1-form symmetry

We stated Poincare duality in euclidean space above and it is straight forward to generalize to the euclidean space-time $d = D$. The line operator here is generalized to a *defect* line, which is an operator extended along time direction.

We can derive the corresponding Ward identities. Consider the correlation function involving a single defect,

$$\langle W[\mathcal{C}]\rangle = \int \mathcal{D}\phi W[\mathcal{C}] e^{iS[\phi]} \tag{49}$$

$$= \int \mathcal{D}\phi' W'[\mathcal{C}] e^{iS[\phi]} \tag{50}$$

$$= \int \mathcal{D}\phi \left(W[\mathcal{C}] + \int_{\mathcal{C}} \xi_1 \delta W[\mathcal{C}] \right) (1 + i\delta S) e^{iS[\phi]}, \tag{51}$$

With δS given in eq.(40), which reads

$$\delta S = \int d^D x \frac{1}{2} J^{\mu\nu} \left(\partial_\mu \xi_\nu - \partial_\nu \xi_\mu\right) = -\int d^D x \xi_\nu \partial_\mu J^{\mu\nu}. \tag{52}$$

The above equations imply

$$i \int d^D x \xi_\nu(x) \langle \partial_\mu J^{\mu\nu} W[\mathcal{C}]\rangle = \int_{\mathcal{C}} dy^\nu \xi_\nu(y) \langle \delta W[\mathcal{C}]\rangle \tag{53}$$

For arbitrary $\xi_\nu(x)$, it follows that

$$\langle \partial_\mu J^{\mu\nu}(x) W[\mathcal{C}]\rangle = -i \int_{\mathcal{C}} dy^\nu \delta^{(D)}(x-y) \langle \delta W[\mathcal{C}]\rangle. \tag{54}$$

5 Examples of 1-form symmetries

5.1 1-form symmetry in $D = 3$ space-time

In Minkovskian 3-dimensional space-time, the conservation law reads

$$\partial_\mu J^{\mu\nu} = \partial_0 J^{0\nu} + \partial_1 J^{1\nu} + \partial_2 J^{2\nu} = 0. \tag{55}$$

It follows that 2 charges

$$Q^1 = \int dx^2 J^{01} \quad \text{and} \quad Q^2 = \int dx^1 J^{02} \tag{56}$$

defined in spatial 1-d curve are conserved and coordinate independent respectively.

$$\frac{dQ^i}{dt} = 0, \qquad \frac{\partial Q^i}{\partial x^i} = 0 \qquad i = 1, 2. \tag{57}$$

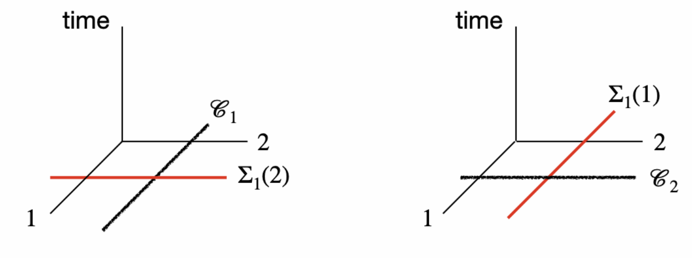

Figure 2: Two lines at a fixed time intersecting in space

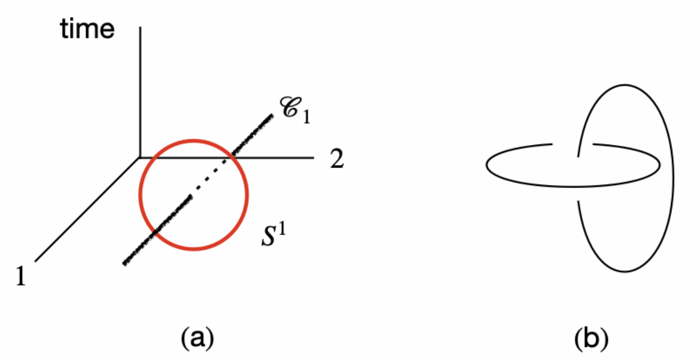

Figure 3: General intersecting of defects

The presence of a line operator $W[\mathcal{C}]$ may violate such symmetries by ward identity eq.(54). For $i = 1$

$$\langle \partial_0 J^{01}(x) W \rangle + \langle \partial_2 J^{21}(x) W \rangle = -i \int_{\mathcal{C}} dy^1 \delta^{(3)}(x-y) \langle \delta W \rangle. \tag{58}$$

Integrating Σ along x^2 direction and over time direction x^0 form $y^0 - \epsilon$ to $y^0 - \epsilon$, leads to

$$\langle [Q^1, W] \rangle = -i \int_{\Sigma} dx^2 \int_{\mathcal{C}} dy^1 \delta^{(2)}(x-y) \langle \delta W \rangle. \tag{59}$$

The intersection of Σ and $\mathcal{C}$ implies non-trivial action of Q^1 acting on line operator $W[\mathcal{C}]$.

We can also assign a topological meaning to the conservation law. We integrate ward identity over a general 2-d manifold Ω_2 in space-time with the boundary curve S^1. By Hodge dual we know that integration element $(d\Omega_2)_\nu$ carries one index ν in 3-d space-time.

$$\int_{\Omega_2} (d\omega_2)_\nu \langle \partial_\mu J^{\mu\nu} W \rangle = -i \int_{\Omega_2} (d\omega_2)_\nu \int_{\mathcal{C}} dy^\nu \delta^{(3)}(x-y) \langle \delta W \rangle. \tag{60}$$

Which can be expressed as

$$\langle Q(S^1) W[\mathcal{C}] \rangle = -i \mathrm{Link}(S^1, \mathcal{C}) \langle \delta W[\mathcal{C}] \rangle. \tag{61}$$

5.2 1-form symmetry in $D = 4$ Maxwell theory

Next we consider an explicit physical theory with 1-form symmetries, which is the free Maxwell theory. The action is defined in terms of a compact $U(1)$ gauge field A with field strength $F = dA$.

$$S[A] = -\frac{1}{4e^2} \int d^4x F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2e^2} \int F \wedge \star F, \tag{62}$$

Where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The compactness nature of $U(1)$ indicates electric charges q are quantized (we omit the proof here). The physical observable of theory (62) must be gauge invariant objects.

One apparent candidate is *the Wilson loop operator*

$$W_q[\mathcal{C}] \equiv \exp\left(iq \oint_{\mathcal{C}} A\right).$$

(63)

The equations of motion of (62) are

$$\frac{1}{e^2} d \star F = 0 \quad \text{and} \quad dF = 0. \tag{64}$$

In components

$$\frac{1}{e^2} \partial_\mu F^{\mu\nu} = 0 \quad \text{and} \quad \epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0. \tag{65}$$

The equations of motion imply that the theory has two types of 1-form symmetries, a 1-form electric and a 1-form magnetic symmetries. The currents are defined

$$J_e \equiv \frac{1}{e^2} F \quad \text{and} \quad J_m \equiv \frac{1}{2\pi} F, \tag{66}$$

respectively. The corresponding charges are

$$Q_e(\Sigma_2) = \int_{\Sigma_2} \star J_e = \frac{1}{e^2} \int_{\Sigma_2} \star F \tag{67}$$

and

$$Q_m(\Sigma_2) = \int_{\Sigma_2} \star J_m = \frac{1}{2\pi} \int_{\Sigma_2} F \tag{68}$$

One word for Maxwell theory in other dimensions. The electric symmetry is always a 1-form symmetry since Q_e integrates over a D-2 closed manifold. However, the magnetic symmetry would become a D-3 form symmetry and integrated over 2-d surface.

The charged operators of electric symmetry is naturally the Wilson loop. To see the charged operator of magnetic symmetry, we notice that in a free Maxwell theory with no electric charge, we can define a magnetic gauge field $\tilde{a}$. Since $d \star F = 0$ and second cohomology class $H^2(\mathbb{R}^4) = 0$. Then $\star F = d\tilde{a}$ for some 1-form $\tilde{a}$ ($D-3$ in D-dim). The action (62) can be written as

$$S[\tilde{a}] = -\frac{e^2}{4} \int d^4x (\star F)^{\mu\nu} (\star F)_{\mu\nu} = -\frac{e^2}{2} \int d\tilde{a} \wedge \star d\tilde{a} \tag{69}$$

The action is invariant under magnetic gauge transform

$$\tilde{a} \to \tilde{a} + d\lambda. \tag{70}$$

In addition to local gauge invariant field $\star F$, we can also construct *'t Hooft line operator* that supported on space-time curve Γ.

$$T_{q_m}[\Gamma] \equiv \exp\left(2\pi i q_m \oint_\Gamma \tilde{a}\right). \tag{71}$$

Here q_m is the magnetic charge, and 't Hooft lines are natural candidates to be charged under magnetic symmetry.